# Dot Product

**Goal:** Define an algebraic operation to capture the geometric relationship between two vectors.

**Def$^n$:** The dot product of vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is $\vec{u} \cdot \vec{v} := u_1 v_1 + u_2 v_2 + u_3 v_3$.

The dot product has many useful algebraic properties.

**Prop:** If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ are vectors and $a \in \mathbb{R}$, then...

⓪ (zero) $\vec{0} \cdot \vec{u} = 0u_1 + 0u_2 + 0u_3 = 0$

① (Commutativity) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = v_1 u_1 + v_2 u_2 + v_3 u_3 = \vec{v} \cdot \vec{u}$

② (scalar factor) $(a\vec{u}) \cdot \vec{v} = a(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (a\vec{v})$.

$(a\vec{u}) \cdot \vec{v} = \langle au_1, au_2, au_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = au_1 v_1 + au_2 v_2 + au_3 v_3 = a(\vec{u} \cdot \vec{v})$

Similar computation $\rightsquigarrow$ $\vec{u} \cdot (a\vec{v}) = a(\vec{u} \cdot \vec{v})$

③ (distribution) $\vec{u} \cdot (\vec{v} + \vec{w}) =$

$\vec{u} \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle = u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3)$

$= (u_1 v_1 + u_2 v_2 + u_3 v_3) + (u_1 w_1 + u_2 w_2 + u_3 w_3) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$

④ (norm) $\vec{v} \cdot \vec{v} = |\vec{v}|^2$

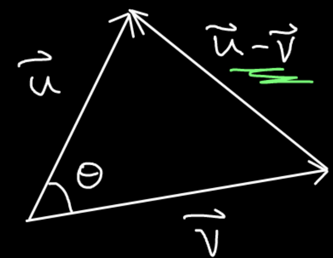
$\vec{v} \cdot \vec{v} = v_1 v_1 + v_2 v_2 + v_3 v_3 = v_1^2 + v_2^2 + v_3^2 = \left(\sqrt{v_1^2 + v_2^2 + v_3^2}\right)^2$

Now we relate the dot product to simple geometry.

**Prop:** If vectors $\vec{u}$ and $\vec{v}$ are at angle $\theta$, then

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos(\theta).$$

**pf:** Consider the triangle formed by $\vec{u}$ and $\vec{v}$. By the Law of Cosines we obtain




$$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(\theta)$$

By the basic properties of the dot product we obtain

$$|\vec{u}-\vec{v}|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$
$$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - 2\,\vec{u}\cdot\vec{v} = |\vec{u}|^2 + |\vec{v}|^2 - 2\,\vec{u}\cdot\vec{v}$$

Hence we obtain the equality

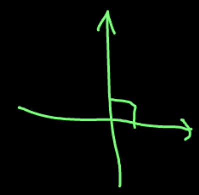
$$|\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u}\cdot\vec{v} = |\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(\theta).$$

Finally, subtracting $|\vec{u}|^2 + |\vec{v}|^2$ from both sides and then dividing by $-2$ we obtain the desired result. ▨

**NB:** We can use this relationship to compute the angle between two vectors $\vec{u}$ and $\vec{v}$. ☺

**Defn:** Vectors $\vec{u}$ and $\vec{v}$ are <u>orthogonal</u> when $\vec{u}\cdot\vec{v} = 0$.

Thus vectors $\vec{u}$ and $\vec{v}$ are orthogonal when



$$0 = \vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$$

$$\iff \quad 0 = |\vec{u}| \quad \text{OR} \quad 0 = |\vec{v}| \quad \text{OR} \quad 0 = \cos(\theta)$$

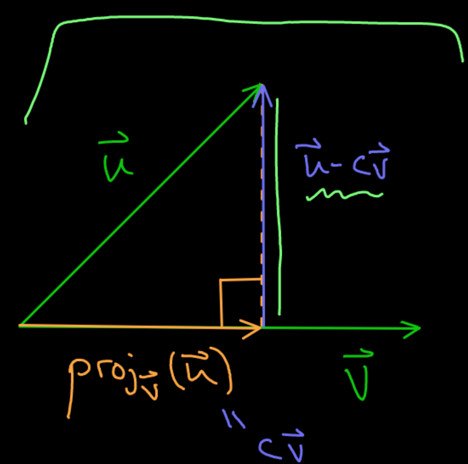
$$\iff \quad \vec{0} = \vec{u} \quad \text{OR} \quad \vec{0} = \vec{v} \quad \text{OR} \quad \theta = \arccos(0)$$

$$\iff \quad \left(\vec{u} = \vec{0} \quad \text{OR} \quad \vec{v} = \vec{0} \quad \text{OR} \quad \theta = \frac{\pi}{2}\right)$$

Thus ① the zero vector is orthogonal to every vector,
and ② nonzero vectors are orthogonal when they're perpendicular.

## ORTHOGONAL PROJECTION

Given two nonzero vectors, we can compute the "component of one in the direction of the other." We do so by projecting at a right angle.




This projection is a scalar multiple of $\vec{v}$. Thus

$\text{proj}_{\vec{v}}(\vec{u}) = c\vec{v}$ must satisfy the equation:

"Vector projection of $\vec{u}$ onto $\vec{v}$"

$$\vec{v}\cdot(\vec{u} - c\vec{v}) = 0$$

$$\iff \vec{v}\cdot\vec{u} - c\,\vec{v}\cdot\vec{v} = 0 \iff \vec{u}\cdot\vec{v} = c\,\vec{v}\cdot\vec{v} = |\vec{v}|^2 \iff c = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}$$

$$\therefore \text{ We have } \text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\,\frac{\vec{v}}{|\vec{v}|}$$

$\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$: signed distance from line of $\vec{v}$ to head of $\vec{u}$; "Scalar projection of $\vec{u}$ onto $\vec{v}$"

Ex: Compute the projection of $\vec{u} = \langle 3,-1,1\rangle$ onto $\vec{v} = \langle 4,7,-4\rangle$.

Sol: $\text{proj}_{\vec{v}}(\vec{u}) = \text{Comp}_{\vec{v}}(\vec{u}) \dfrac{\vec{v}}{|\vec{v}|}$ ← unit vector; scalar projection

$\vec{u}\cdot\vec{v} = \langle 3,-1,1\rangle \cdot \langle 4,7,-4\rangle = 3\cdot 4 + -1\cdot 7 + 1\cdot -4 = 12-7-4 = 1$

$|\vec{v}| = \sqrt{4^2+7^2+(-4)^2} = \sqrt{16+49+16} = \sqrt{81} = 9$

$\text{Comp}_{\vec{v}}(\vec{u}) = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \dfrac{1}{9}$ and $\dfrac{1}{|\vec{v}|}\vec{v} = \left\langle \dfrac{4}{9}, \dfrac{7}{9}, \dfrac{-4}{9}\right\rangle$

$\therefore \text{proj}_{\vec{v}}(\vec{u}) = \text{Comp}_{\vec{v}}(\vec{u}) \dfrac{1}{|\vec{v}|}\vec{v} = \dfrac{1}{9}\left\langle \dfrac{4}{9}, \dfrac{7}{9}, \dfrac{-4}{9}\right\rangle = \left\langle \dfrac{4}{81}, \dfrac{7}{81}, \dfrac{-4}{81}\right\rangle$. //

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In $\mathbb{R}^3$, we can use the dot product to compute angles to the standard axes. Let $\alpha$, $\beta$, and $\gamma$ denote the angles from a given vector $\vec{v}$ to the positive $x$-, $y$-, and $z$- axes respectively. Note that these axes are represented by the vectors $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ respectively.

If $\vec{v} = \langle x_0, y_0, z_0\rangle$, then we obtain

$$\vec{\imath}\cdot\vec{v} = \langle 1,0,0\rangle \cdot \langle x_0,y_0,z_0\rangle = x_0 \quad \text{and}$$

$$\vec{\imath}\cdot\vec{v} = |\vec{\imath}||\vec{v}|\cos(\alpha) = \sqrt{1^2+0^2+0^2}\cdot|\vec{v}|\cos(\alpha) = |\vec{v}|\cos(\alpha).$$

Hence when $\vec{v} \neq \vec{0}$ we obtain $\cos(\alpha) = \dfrac{x_0}{|\vec{v}|}$. Similarly we obtain $\cos(\beta) = \dfrac{y_0}{|\vec{v}|}$ and $\cos(\gamma) = \dfrac{z_0}{|\vec{v}|}$. Finally notice

$$\frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{x_0}{|\vec{v}|}, \frac{y_0}{|\vec{v}|}, \frac{z_0}{|\vec{v}|}\right\rangle = \langle \cos(\alpha), \cos(\beta), \cos(\gamma)\rangle.$$

# Examples

**Ex:** When is $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|$?

**Sol:** $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$ is always true.

So if $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|$, then $|\vec{u}||\vec{v}| = |\vec{u}||\vec{v}|\cos(\theta)$.

If $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$, $\vec{u}\cdot\vec{v} = 0 = |\vec{u}||\vec{v}|$.

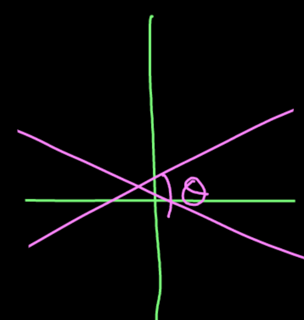
otherwise $1 = \cos(\theta)$, i.e. $\theta = \arccos(1) = 0$.

Hence $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|$ if and only if either $\vec{u} = 0$ or $\vec{v} = 0$ or $\vec{u}$ and $\vec{v}$ have the same unit vector. ///

**Ex:** Compute the accute angle between the lines

$$2x - 3y = 3 \quad \text{and} \quad 3x + y = 7.$$

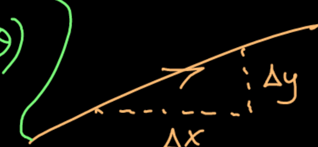
**Sol:** The direction of the lines are determined by:




$2x - 3y = 3 \iff y = \frac{-2}{-3}x + 3 \rightsquigarrow \vec{u} = \langle -3, -2 \rangle$

$3x + y = 7 \iff y = \frac{-3}{1}x + 7 \rightsquigarrow \vec{v} = \langle 1, -3 \rangle$

not to scale




$\therefore$ the desired angle satisfies $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$

$\vec{u}\cdot\vec{v} = \langle -3, -2 \rangle \cdot \langle 1, -3 \rangle = -3 + 6 = 3$

$|\vec{u}| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}$

$|\vec{v}| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$

$\therefore \quad 3 = \sqrt{13}\sqrt{10}\cos(\theta) \iff \cos(\theta) = \frac{3}{\sqrt{130}} \iff \theta = \arccos\left(\frac{3}{\sqrt{130}}\right)$ ///

Ex: Are the given vectors orthogonal, parallel, or neither?

ⓐ $\vec{u} = \langle -5, 4, -2 \rangle$ and $\vec{v} = \langle 3, 4, -1 \rangle$

$\vec{u} \cdot \vec{v} = \langle -5, 4, -2 \rangle \cdot \langle 3, 4, -1 \rangle = -15 + 16 + 2 = 3$ So not orthogonal.

$\frac{\vec{u}}{|\vec{u}|} = \frac{1}{\sqrt{(-5)^2 + 4^2 + (-2)^2}} \langle -5, 4, -2 \rangle = \frac{1}{\sqrt{45}} \langle -5, 4, -2 \rangle = \langle -\frac{5}{3\sqrt{5}}, \frac{4}{3\sqrt{5}}, -\frac{2}{3\sqrt{5}} \rangle$

$\frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{3^2 + 4^2 + (-1)^2}} \langle 3, 4, -1 \rangle = \frac{1}{\sqrt{26}} \langle 3, 4, -1 \rangle = \langle \frac{3}{\sqrt{26}}, \frac{4}{\sqrt{26}}, -\frac{1}{\sqrt{26}} \rangle$

$\neq \pm$ each other, not parallel!

$\therefore$ neither.

ⓑ $9\vec{i} - 6\vec{j} + 3\vec{k}$ and $-6\vec{i} + 4\vec{j} - 2\vec{k}$

$\vec{u} = \langle 9, -6, 3 \rangle$ $\quad \vec{v} = \langle -6, 4, -2 \rangle$

$\vec{u} \cdot \vec{v} = -54 - 24 - 6 = -84 \neq 0$ so not orthogonal.

$\frac{\vec{u}}{|\vec{u}|} = \frac{1}{\sqrt{9^2 + 6^2 + 3^2}} \langle 9, -6, 3 \rangle = \frac{1}{3\sqrt{3^2 + 2^2 + 1^2}} \langle 9, -6, 3 \rangle = \frac{1}{\sqrt{14}} \langle 3, -2, 1 \rangle$

$\frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{6^2 + 4^2 + 2^2}} \langle -6, 4, -2 \rangle = \frac{1}{2\sqrt{3^2 + 2^2 + 1^2}} \langle -6, 4, -2 \rangle = -\frac{1}{\sqrt{14}} \langle 3, -2, 1 \rangle$

Hence $\vec{u}$ is parallel to $\vec{v}$ ◪

Ex: Which of these expressions is meaningful? Why (not)?

ⓐ $(\vec{u} \cdot \vec{v}) \cdot \vec{w}$

scalar dot vect

Not meaningful! (can't dot vectors w/ scalars)

ⓑ $(\vec{u} \cdot \vec{v}) \vec{w}$

scalar vector

Is meaningful! (scalar multiple)

ⓒ $|\vec{u}| (\vec{v} \cdot \vec{w})$

scalar scalar

Is meaningful! (scalar product)

ⓓ $\vec{u} \cdot (\vec{v} + \vec{w})$

vector dot vector

Is meaningful! (vector dot vector)

ⓔ $\vec{u} \cdot \vec{v} + \vec{w}$

scalar plus vector

Not meaningful! (can't add scalars to vectors!)

ⓕ $|\vec{u}| \cdot (\vec{v} + \vec{w})$

scalar dot vector

Not meaningful! (can't dot vectors w/ scalars) ◪

Ex: Compute the angle between the given vectors.

(a) $\vec{u} = \langle -2, 5 \rangle$, $\vec{v} = \langle 5, 12 \rangle$

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$

$\iff \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \cos(\theta)$

$\iff \theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$

$$\theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \arccos\left(\frac{-10+60}{\sqrt{4+25}\sqrt{25+144}}\right)$$

$$= \arccos\left(\frac{50}{\sqrt{29}\sqrt{169}}\right) = \arccos\left(\frac{50}{13\sqrt{29}}\right)$$

(b) $\vec{u} = \langle -1, 3, 4 \rangle$, $\vec{v} = \langle 5, 2, 1 \rangle$

$\vec{u}\cdot\vec{v} = -5 + 6 + 4 = 5$

$|\vec{u}| = \sqrt{1+9+16} = \sqrt{26}$

$|\vec{v}| = \sqrt{25+4+1} = \sqrt{30}$

$\therefore \theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \arccos\left(\frac{5}{\sqrt{26}\sqrt{30}}\right)$

$$= \arccos\left(\frac{5}{2\sqrt{13}\cdot\sqrt{15}}\right)$$

(c) $\vec{u} = 4\vec{i} - 3\vec{j} + \vec{k}$, $\vec{v} = 2\vec{i} - \vec{k}$

$\vec{u} = \langle 4, -3, 1 \rangle$, $\vec{v} = \langle 2, 0, -1 \rangle$

$\vec{u}\cdot\vec{v} = 8 + 0 - 1 = 7$

$|\vec{u}| = \sqrt{16+9+1} = \sqrt{26}$

$|\vec{v}| = \sqrt{4+0+1} = \sqrt{5}$

$\therefore \theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \arccos\left(\frac{7}{\sqrt{130}}\right)$

▨

Ex: Compute the scalar and vector projections of $\vec{u}$ on $\vec{v}$ (1) and $\vec{v}$ on $\vec{u}$ (2) for $\vec{u} = \langle -1, 4, 8 \rangle$ and $\vec{v} = \langle 12, 1, 2 \rangle$.

$$\text{comp}_{\vec{b}}(\vec{a}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}, \quad \text{proj}_{\vec{b}}(\vec{a}) = \text{comp}_{\vec{b}}(\vec{a})\,\frac{\vec{b}}{|\vec{b}|}$$

$\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u} = -12 + 4 + 16 = 8$

$|\vec{u}| = \sqrt{1+16+64} = \sqrt{81} = 9$

$|\vec{v}| = \sqrt{144+1+4} = \sqrt{149}$

$\text{proj}_{\vec{v}}(\vec{u}) \neq \text{proj}_{\vec{u}}(\vec{v})$

(1) $\text{comp}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \frac{8}{\sqrt{149}}$ and $\text{proj}_{\vec{v}}(\vec{u}) = \frac{8}{\sqrt{149}}\cdot\frac{1}{\sqrt{149}}\langle 12, 1, 2 \rangle = \frac{8}{149}\langle 12, 1, 2 \rangle$

(2) $\text{comp}_{\vec{u}}(\vec{v}) = \frac{\vec{v}\cdot\vec{u}}{|\vec{u}|} = \frac{8}{9}$ and $\text{proj}_{\vec{u}}(\vec{v}) = \frac{8}{9}\cdot\frac{1}{9}\langle -1, 4, 8 \rangle = \frac{8}{81}\langle -1, 4, 8 \rangle$

▨

Ex: Compute the direction cosines and angles of the vector $\vec{v} = \langle 6, 3, -2 \rangle$.

Sol: From the lecture $\frac{\vec{v}}{|\vec{v}|} = \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$.

$$\frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{36+9+4}} \langle 6, 3, -2 \rangle = \frac{1}{\sqrt{49}} \langle 6, 3, -2 \rangle = \left\langle \frac{6}{7}, \frac{3}{7}, -\frac{2}{7} \right\rangle$$

$$\therefore \underbrace{\begin{cases} \cos(\alpha) = \frac{6}{7} \\ \cos(\beta) = \frac{3}{7} \\ \cos(\gamma) = -\frac{2}{7} \end{cases}}_{\text{direction cosines}} \qquad \therefore \underbrace{\begin{cases} \alpha = \arccos\left(\frac{6}{7}\right) \\ \beta = \arccos\left(\frac{3}{7}\right) \\ \gamma = \arccos\left(-\frac{2}{7}\right) \end{cases}}_{\text{direction angles}}$$

$\boxtimes$

Ex: Show any two direction angles determine the third in $\mathbb{R}^3$.

Sol: We know $\frac{\vec{v}}{|\vec{v}|} = \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$, so

$$1^2 = \left| \frac{\vec{v}}{|\vec{v}|} \right|^2 = \left| \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle \right|^2$$

$$= \cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma)$$

$\therefore$ any two direction cosines determines the third!! $\boxtimes$

Ex: Compute the distance from point $(x_0, y_0)$ to line $ax + by + c = 0$.

Sol:

Prop: Let $\vec{u}$ and $\vec{v}$ be vectors. The vector $|\vec{v}|\vec{u} + |\vec{u}|\vec{v}$ bisects the angle between $\vec{u}$ and $\vec{v}$.

pf:

Prop (Cauchy-Schwarz Inequality): Every pair of vectors $\vec{u}, \vec{v}$ satisfies $|\vec{u} \cdot \vec{v}| \leq |\vec{u}||\vec{v}|$.

pf:

<u>Prop (Triangle Inequality)</u>: For every pair of vectors $\vec{u}, \vec{v}$ we have $|\vec{u} + \vec{v}| \le |\vec{u}| + |\vec{v}|$.

<u>pf</u>:

**Prop (Parallelogram Law):** Every pair of vectors $\vec{u}, \vec{v}$ satisfies

$$|\vec{u}+\vec{v}|^2 + |\vec{u}-\vec{v}|^2 = 2|\vec{u}|^2 + 2|\vec{v}|^2.$$

pf: